OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4726/01**

MATHEMATICS

Further Pure Mathematics 2

FRIDAY 23 MAY 2008 Morning

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

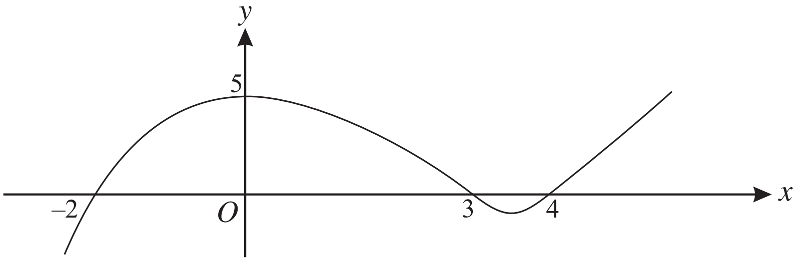
- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

[Turn over

1 It is given that $f(x) = \dfrac{2ax}{(x-2a)(x^2+a^2)}$, where a is a non-zero constant. Express $f(x)$ in partial fractions. [5]

2



The diagram shows the curve $y = f(x)$. The curve has a maximum point at (0, 5) and crosses the x-axis at (−2, 0), (3, 0) and (4, 0). Sketch the curve $y^2 = f(x)$, showing clearly the coordinates of any turning points and of any points where this curve crosses the axes. [5]

3 By using the substitution $t = \tan \frac{1}{2}x$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2-\cos x}\, dx,$$

giving the answer in terms of π. [6]

4 **(i)** Sketch, on the same diagram, the curves with equations $y = \operatorname{sech} x$ and $y = x^2$. [3]

(ii) By using the definition of $\operatorname{sech} x$ in terms of e^x and e^{-x}, show that the x-coordinates of the points at which these curves meet are solutions of the equation

$$x^2 = \frac{2e^x}{e^{2x}+1}.$$ [3]

(iii) The iteration

$$x_{n+1} = \sqrt{\frac{2e^{x_n}}{e^{2x_n}+1}}$$

can be used to find the positive root of the equation in part **(ii)**. With initial value $x_1 = 1$, the approximations $x_2 = 0.8050$, $x_3 = 0.8633$, $x_4 = 0.8463$ and $x_5 = 0.8513$ are obtained, correct to 4 decimal places. State with a reason whether, in this case, the iteration produces a 'staircase' or a 'cobweb' diagram. [2]

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x \, dx.$$

(i) By considering $I_n + I_{n-2}$, or otherwise, show that, for $n \geqslant 2$,

$$(n-1)(I_n + I_{n-2}) = 1.$$ [4]

(ii) Find I_4 in terms of π. [4]

6 It is given that $f(x) = 1 - \dfrac{7}{x^2}$.

(i) Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 6 decimal places. [3]

(ii) The root of $f(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α. [1]

(iii) The error e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers correct to 5 decimal places. Verify that $e_3 \approx \dfrac{e_2^3}{e_1^2}$. [3]

7 It is given that $f(x) = \tanh^{-1}\left(\dfrac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$.

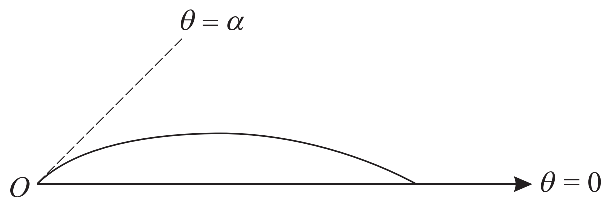
(i) Show that $f'(x) = -\dfrac{1}{1+2x}$, and find $f''(x)$. [6]

(ii) Show that the first three terms of the Maclaurin series for $f(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found. [4]

8 The equation of a curve, in polar coordinates, is

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

(i)



The diagram shows the part of the curve for which $0 \leqslant \theta \leqslant \alpha$, where $\theta = \alpha$ is the equation of the tangent to the curve at O. Find α in terms of π. [2]

(ii) (a) If $f(\theta) = 1 - \sin 2\theta$, show that $f\left(\frac{1}{2}(2k+1)\pi - \theta\right) = f(\theta)$ for all θ, where k is an integer. [3]

(b) Hence state the equations of the lines of symmetry of the curve

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$ [2]

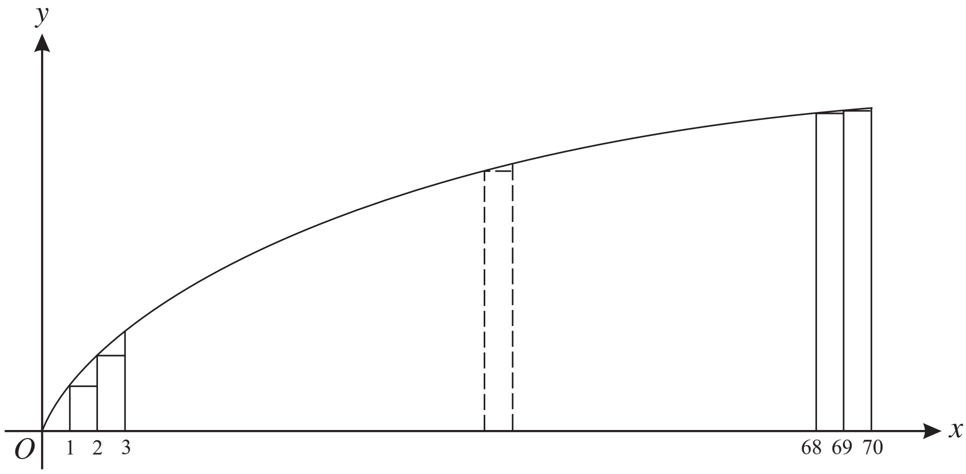
(iii) Sketch the curve with equation

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

State the maximum value of r and the corresponding values of θ. [4]

9 **(i)** Prove that $\int_0^N \ln(1+x)\,dx = (N+1)\ln(N+1) - N$, where N is a positive constant. [4]

(ii)



The diagram shows the curve $y = \ln(1+x)$, for $0 \leqslant x \leqslant 70$, together with a set of rectangles of unit width.

(a) By considering the areas of these rectangles, explain why

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 < \int_0^{70} \ln(1+x)\,dx.$$ [2]

(b) By considering the areas of another set of rectangles, show that

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 > \int_0^{69} \ln(1+x)\,dx.$$ [3]

(c) Hence find bounds between which $\ln(70!)$ lies. Give the answers correct to 1 decimal place. [3]

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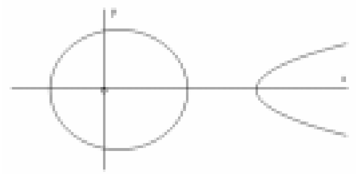
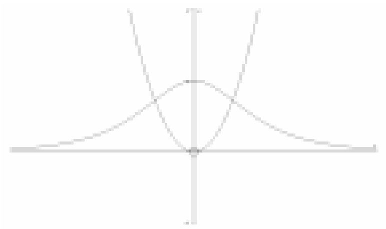
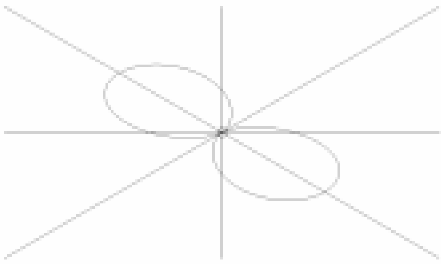
Q		Answer	Mark	Guidance
1		Write as $\dfrac{A}{x-2a}+\dfrac{Bx+C}{x^2+a^2}$	M1	Accept C=0
		Get $2ax = A(x^2+a^2) + (Bx+C)(x-2a)$	A1√	Follow-on for C=0
		Choose values of x and/or equate coeff.	M1	Must lead to at least one of their A,B,C
		Get $A = {}^4/_5$, $B = -{}^4/_5$, $C = {}^2/_5 a$	A1	For two correct from correct working only
			A1	For third correct
			5	
2			B1	Get (4,0), (3,0), (-2,0) only
			B1	Get $(0,\sqrt{5})$ as "maximum"
			B1	Meets x-axis at 90^0 at all crossing points
			B1	Use $-2\le x\le 3$ and $x\ge 4$ only
			B1	Symmetry in Ox
			5	
3		Quote/derive $\mathrm{d}x = \dfrac{2}{1+t^2}\,\mathrm{d}t$	B1	
		Replace all x and dx from their expressions	M1	Not dx=dt; ignore limits
		Tidy to $2/(3t^2+1)$	A1	Not $a/(3t^2+1)$
		Get $k\tan^{-1}(At)$	M1	Allow A=1 if from $p/(t^2+1)$ only
		Get $k = ⅔\sqrt{3}$, $A = \sqrt{3}$	A1√	Allow $k=a/\sqrt{3}$ from line 3; AEEF
		Use limits correctly to ${}^2/_9\sqrt{3}\pi$	A1	AEEF
			6	
4	(i)		B1	Correct $y = x^2$
			B1	Correct shape/asymptote
			B1	Crossing (0,1)
			3	
	(ii)	Define sech $x = 2/(e^x + e^{-x})$	B1	AEEF
		Equate their expression to x^2 and attempt to simplify	M1	
		Clearly get A.G.	A1	
			3	
	(iii)	Cobweb	B1	
		Values > and then < root	B1	Only from cobweb
			2	

5	(i)	Factorise to $\tan^{n-2}x(1+\tan^2 x)$	**B1**	Or use $\tan^n x = \tan^{n-2}x.\tan^2 x$
		Clearly use $1+\tan^2 = \sec^2$	**M1**	Allow wrong sign
		Integrate to $\tan^{n-1}x/(n-1)$	**A1**	Quote or via substitution
		Use limits and tidy to A.G.	**A1**	Must be clearly derived
			4	
	(ii)	Get $3(I_4 + I_2) = 1$, $I_2 + I_0 = 1$	**B1**	Write down one correct from reduction formula
		Attempt to evaluate I_0 (or I_2)	**M1**	$I_2 = a\tan x + b$, $a,b \neq 0$
		Get $\frac{1}{4}\pi$ (or $1 - \frac{1}{4}\pi$)	**A1**	
		Replace to $\frac{1}{4}\pi - \frac{2}{3}$	**A1**	
			4	
6	(i)	Attempt to use N-R of correct form with clear $f'(x)$ used	**M1**	
		Get 2.633929, 2.645672	**A1**	For one correct to minimum of 6 d.p.
			A1√	For other correct from their x_2 in correct NR
			3	
	(ii)	$\sqrt{7}$	**B1**	Allow ±
			1	
	(iii)	Get $e_1 = 0.14575$, $e_2 = 0.01182$	**B1**√	From their values
		Get $e_3 = 0.00008$	**B1**√	
		Verify both ≈ 0.00008	**B1**	From 0.000077.. or $0.01182^3/0.14575^2$
			3	
7	(i)	Attempt quotient/product on bracket	**M1**	
		Get $-3/(2+x)^2$	**A1**	May be implied
		Use Formulae Booklet or derive from $\tanh y = (1-x)/(2+x)$	**M1**	Attempt $\tanh^{-1}$ part in terms of x
		Get $\frac{-3}{(2+x)^2} \cdot \frac{1}{1-((1-x)/(2+x))^2}$	**A1**√	From their results above
		Clearly tidy to A.G.	**A1**	
		Get $f''(x) = 2/(1+2x)^2$	**B1**	cao
			6	
			SC	Use reasonable ln definition **M1**
				Get $y = \frac{1}{2}\ln((1-k)/(1+k))$ for $k = (1-x)/(1+2x)$ **A1**
				Tidy to $y = \frac{1}{2}\ln(3/(1+2x))$ **A1**
				Attempt chain rule **M1**
				Clearly tidy to A.G. **A1**
				Get $f''(x)$ **B1**
	(ii)	Attempt $f(0)$, $f'(0)$ and $f''(0)$	**M1**	From their differentiation
		Get $\tanh^{-1}\frac{1}{2}$, -1 and 2	**A1**√	
		Replace $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln 3$ ($=\ln\sqrt{3}$)	**B1**	Only
		Get $\ln\sqrt{3} - x + x^2$	**A1**	
			4	
			SC	Use standard expansion from $\frac{1}{2}\ln 3 - \frac{1}{2}\ln(1+2x)$

8 (i)	Attempt to solve $r = 0$	**M1**	
	Get $\alpha = \frac{1}{4}\pi$	**A1**	From correct method; ignore others; allow θ
		2	
(ii) (a)	Get $1 - \sin((2k+1)\pi - 2\theta)$	**M1**	Attempt $f(\frac{1}{2}(2k+1)\pi - \theta)$, leading to 2θ here
	Expand as $\sin(A+B)$	**M1**	Or discuss periodicity for general k
	Use k as integer so $\sin(2k+1)\pi = 0$,		
	And $\cos(2k+1)\pi = -1$	**A1**	Needs a clear explanation
		3	
(b)	Quote $\frac{1}{4}(2k+1)\pi$	**B1**	For general answer or 2 correct (ignore other answers given)
	Select or give $k = 0,1,2,3$	**B1**	For all 4 correct in $0 \le \theta < 2\pi$
		2	
(iii)		**B1**	Correct shape; 2 branches only, roughly as shown
		B1	Clear symmetry in correct rays
		B1	Get max. $r = 2$
		B1	At $\theta = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$; both required (allow correct answers not in $0 \le \theta < 2\pi$ here)
		4	
9 (i)	Attempt to use parts	**M1**	Two terms, one yet to be integrated
	Divide out $x/(1+x)$	**M1**	Or use substitution
	Correct answer $x\ln(1+x) - x + \ln(1+x)$	**A1**	
	Limits to correct A.G.	**A1**	
		4	
		SC	Quote $\int \ln x \, dx$ **M1**
			Clear use of limits to A.G. **A1**
		SC	Attempt to diff'ate by product rule **M1**
			Clear use of limits to A.G. **A1**
(ii) (a)	Use sum of areas of rect.< Area under curve (between limits 0 and 70)	**B1**	
	Areas = 1x heights = 1(ln2 + ln3+…ln70)	**B1**	<u>Areas</u> to be specified
		2	
(b)	Explain use of 69	**B1**	Allow diagram or use of left shift of 1 unit
	Explain first rectangle	**B1**	
	Areas as above > area under curve	**B1**	
		3	
(c)	Show/quote ln2 + ln3 + … ln70 = ln 70!	**B1**	
	Use N = 69, 70 in (i)	**M1**	No other numbers; may be implied by 228.39.. or 232.65.. seen; allow 228.4, 232.6 or 232.7
	Get 228.3, 232.7	**A1**	
		3	